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# A Note on Flag Curvature in Finsler Space

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**ABSTRACT**: This paper has been devoted to the study of Finsler space with flag curvature. Section 1 is devoted to the study of theory of indicatrix. Section 2 delineates to the metric non-linear connections. Section 3 is devoted to the study of flag curvature in Finsler space.

**KEYWORDS:** Minkowski space, C-projective, Flag curvature, Finsler space.

## I. INTRODUCTION

Let $M^n$ be a Minkowski space with the indicatrix $F(X^i) = 1$, wherein $X^i(i = 1, 2, 3, - - - -, n)$ may be regarded as components of a vector. The indicatrix $F(X^i) = 1$ is given by $X^i = X^i(u^\alpha)$ with n-1 parameters $u^\alpha(\alpha= 1, 2, 3,- - -,n-1)$ that is to say, we shall regard the indicatrix as an (n-1)-dimensional manifold $I^{n-1}$ with coordinate system $(u^\alpha)$. The indicatrix is given by [4]:

(1.1) $F^2(X) = g_{ij}(X) X^iX^j = 1,$

If the Minkowski space $M^n$ be a Riemannian space with the metric tensor given by $g_{ij}(X)$, then the induced metric tensor $g_{\alpha\beta}(u)$ on the indicatrix $I^{n-1}$ is given by [4]:

(1.2) $g_{\alpha\beta} = g_{\alpha\beta}(u) = g_{ij} X^i_\alpha X^j_\beta$

Wherein

(1.3) $X^i_\alpha = (\partial X^i/\partial u^\alpha).$

Differentiating equation (1.1) by $u^\alpha$ and using equation (1.3), we get

(1.4) $g_{ij} X^i_\alpha X^j = 0$

This equation shows that the vector $X^i$ is normal unit vector of $I^{n-1}$.

The covariant derivative of $X^i_\alpha$ and using D-symbol, we get [6]:

(1.5) $D_\alpha X^i_\beta = H^i_{\alpha\beta},$

(1.6) $D_\alpha X^j_\beta = h_{\alpha\beta}X^i$

and

(1.7) $D_\alpha X^i_\beta=\partial_\alpha X^i_\beta+\{^i_{j\,k}\}X^j_\alpha X^k_\beta -\{^\gamma_{\alpha\,\beta}\}X^i_\gamma.$

Differentiating equation (1.4) covariantly and using D-symbol, we obtain

(1.8) $g_{ij}X^i_\alpha X^j_\beta + g_{ij} (D_\beta X^i_\alpha)X^j = 0$

and

(1.9) $g_{\alpha\beta} + h_{\alpha\beta} = 0.$

By virtue of equations (1.2) and (1.8) yields

(1.10) $g_{\alpha\beta} + g_{ij}(D_\beta X^i_\alpha)X^j = 0$

From equations (1.6) and (1.10), we get

(1.11) $g_{\alpha\beta} + g_{ij}\, h_{\beta\alpha}X^iX^j = 0,$

In view of equations (1.1) and (1.11), we obtain

(1.12) $g_{\alpha\beta} + h_{\beta\alpha} = 0,$

Comparing equations (1.9) and (1.12) yields

(1.13) $h_{\alpha\beta} = h_{\beta\alpha},$

Hence, $h_{\alpha\beta}$ is symmetric tensor of $I^{n-1}$.

In view of equations (1.2) and (1.9), we obtain

(1.14) $h_{\alpha\beta} = - g_{ij}X^i_\alpha X^j_\beta$

Contracting equation (1.9) with $g^{\alpha\gamma}$, we obtain

(1.15) $g^{\alpha\gamma}h_{\alpha\beta} = - \delta^\gamma_\beta$

Contracting equation (1.9) with $g^{\alpha\beta}$ yields

(1.16) $g^{\alpha\gamma}h_{\alpha\beta} = -(n-1).$

## II. METRIC NON-LINEAR CONNECTIONS

Consider a differentiable vector field $X^i$ in a Finsler space $F^n$, and there is given a set of functions $\Gamma^{1i}_k(x,X)$ depending on this field, wherein $\Gamma^{1i}_k(x,X)$ are homogeneous of degree one in the $X^i$. Then absolute differential is defined as

(2.1) $\delta X^i = dX^i + \Gamma^{1i}_k(x,X)dx^k$

Next, consider a covariant vector field $Y_i$ in the Finsler space $F^n$. Then the absolute differential of $Y_i$ is defined as

(2.2) $\delta Y_i = dY_i - \Gamma^2_{ik}(x,Y)dx^k$

Wherein $\Gamma^2_{ik}(x,Y)$ is homogeneous of degree one in $Y_i$.

Let us consider a relation between $X^i$ and $Y_i$ is

(2.3) $Y_i = g_{ij}X^j$,

Contracting equation (2.3) by $X^i$ and using equation (1.1), we get

(2.4) $X^i Y_i = 1$,

The absolute differential $\delta Y_i$ coincides with the covariant component of $\delta X^i$, i.e.

(2.5) $\delta Y_i = g_{ij}\, \delta X^j$

Consequently yields

(2.6) $\Gamma^2_{ik}(x,Y) = (\partial g_{ij}/\partial x^k)X^j - g_{ij}\, \Gamma^{1i}_{k}(x,X)$.

Next, we shall find out another form of the coefficients $\Gamma^{1i}_{k}$ and $\Gamma^2_{jk}$. The following equation in $F(x,X)$ is

(2.7) $F_{,k} = (\partial F/\partial x^k) - (\partial F/\partial X^j)G^j_{,k}(x,X) = 0$

Wherein

(2.8) $G^j_{,k}(x,X) = (\partial G^j/\partial X^k)$

and

(2.9) $G^i(x,X) = (1/2)\{^{\,i}_{h\,k}\}X^hX^k$.

From equations (2.1) and (2.7), we have

(2.10) $\Gamma^{1i}_{k}(x,X) = T^i_k(x,X) + G^i_k(x,X)$,

Wherein $T^i_k(x,X)$ is an arbitrary tensor homogeneous of degree one in $X^i$ which satisfies the relations

(2.11) $g_{ij}\, X^i\, T^j_k = 0$

and

(2.12) $B^i_j\, T^j_k = T^i_k$.

Therefore, we take

(2.13) $\Gamma^2_{jk}(x,Y) = T^*_{jk}(x,Y) + G^i_{,jk}(x,\phi(x,Y))Y_i$,

Wherein $T^*_{jk}$ is an arbitrary tensor homogeneous of degree one in $Y_i$ which is restricted by the relation $T^*_{jk}\, X^j = 0$.

In view of equations (2.10) and (2.13), the equation (2.6) assumes the form

(2.14) $T^*_{jk}(x,Y)=-g_{ij}T^i_k+\{(\partial g_{ij}/\partial x^k)X^i-g_{ij}G^i_k-G^i_{,jk}Y_i\}$

But

(2.15) $g_{ij}G^i_{,k}\, X^j = (1/2)(\partial g_{jm}/\partial x^k)X^j\, X^m$
$= g_{ij}X^j\, (\{^{\,i}_{m\,k}\}X^m - \{^{\,l}_{m\,n}\}C^i_{kl}X^m\, X^n)$,

Differentiating equation (2.15) by $X^h$ yields

(2.16) $(\partial g_{hm}/\partial x^k)X^m = g_{ih}G^i_{,k} + Y_iG^i_{,kh}$,

Inserting equation (2.16) in the equation (2.14), we obtain

(2.17) $T^*_{jk} = - g_{ij}\, T^i_k$.

Since

(2.18) $T_{jk} = g_{ij}T^i_k$.

Using equation (3.18) in the equation (2.17), we get

(2.19) $T^*_{jk} = - T_{jk}$.

**Theorem 2.1:**

If the coefficients $\Gamma^{2h}_{jk}$ of a relative connection parameters is symmetric with indices j and k then the tensor $T_{jk}$ is also symmetric with indices j and k.

**Proof:**

In view of equations (2.13) and (2.19), we obtain

(2.20) $\Gamma^2_{jk}(x,Y)=-T_{jk}(x,\phi(x,Y)) + G^i_{,jk}(x,\phi(x,Y))Y_i$,

Interchanging the indices j and k in equation (2.20), we get

(2.21) $\Gamma^2_{kj}(x,Y) = -T_{kj}(x,\phi(x,Y))+G^i_{,kj}(x,\phi(x,Y))Y_i$,

Since $\Gamma^2_{jk}(x,Y)$ is symmetric with j and k, then equation (2.21) reduces in the form

(2.22) $\Gamma^2_{jk}(x,Y) = -T_{kj}(x, \phi(x,Y))+G^i_{,jk}(x, \phi(x,Y))Y_i$,

From equations (2.20) and (2.22), we obtain

(2.23) $T_{jk} = T_{kj}$,

Hence, $T_{jk}$ is symmetric with indices j and k.

## III. FLAG CURVATURE IN FINSLER SPACE

If M is an n-dimension $C^\infty$ space and F is the Finsler metric then we assume that $F^n(M,F)$ be a Finsler space. F is assumed to be a $C^\infty$ function on the slit tangent bundle $TxM^o = TxM\backslash\{0\}$ satisfying the condition:

(a) F is $C^\infty$ on $TxM^o$

(b) $F(x,ky) = k\, F(x,y)$, for any $x\epsilon M, y\epsilon T_xM$ and $k>0$

(c) $g_{ab} = (1/2)\{\partial^2F^2/\partial y^a\partial y^b\}$,

is positive definite at every point (x,y) of $TxM^o$. It is to be noted that $(x^a,y^b)$ are the coordinates on TxM where $(x^a)$ are the coordinates on M. $(\partial/\partial x^a, \partial/\partial y^b)$ is the local fram field on TxM. Then the Liouville vector field

(3.1) $L = y^a(\partial/\partial y^a)$

is defined as a global section of the vertical vector bundle $TxM^o$.

Further,

(3.2) $L = l\, F$

is a unit vector field,

(3.3) $g_{ab}\, l^a\, l^b = 1$,

Wherein

(3.4) $l^a\, F = y^a$.

Now, Let us assume a flag $y\Lambda u$ at $x\epsilon M$ determined by y and $u= u^a(\partial/\partial x^a)$. Flag curvature is first used by L. Berwald [3]. The flag curvature for the flag $y\Lambda u$ is the number[2,1]:

(3.5) $K = (R_{ab}u^au^b)/\{(g_{ab}u^au^b) - (g_{ab}y^au^b)^2\}$.

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Flag curvature K must be constant when the flag curvature K depends neither on $y^a$ nor on $u^a$. Also it is well known that $F^n$ has constant flag curvature K iff

(3.6) $R_{ab} = K\, h_{ab}$,

Wherein $h_{ab}$ are the components of the angular metric on $F^n$ defined by

(3.7) $h_{ab} = g_{ab} - l_a l_b$.

The Riemannian curvature tensor of Berwald connection is defined as

(3.8) $K^a{}_{bcd} = \lambda_c G^a{}_{bd} + G^e{}_{bd} G^a{}_{ec} - \lambda_d G^a{}_{bc} + G^e{}_{bc} G^a{}_{ed}$.

If we take

(3.9) $K^a{}_{bc} = K^a{}_{0bc}$

and

(3.10) $K^a{}_b = K^a{}_{0b}$,

Consequently, yields

(3.11) $K^a{}_{bc} = (1/3)(\dot{\partial}_b K^a{}_c - \dot{\partial}_c K^a{}_b)$.

The projective Weyl curvature is expressed as follows [7,8]:

(3.12) $W^a{}_{bcd} = K^a{}_{bcd} + \{1/(n^2-1)\}\{\delta^a{}_b(\dot{K}_{dc} - \dot{K}_{cd}) + \delta^a{}_d \dot{K}_{bc} - \delta^a{}_c \dot{K}_{bd} + F\, l^a\, \dot{\partial}_b(\dot{K}_{dc} - \dot{K}_{cd})$,

Wherein

(3.13) $\dot{K}_{ab} = nK_{ab} + K_{ba} + F\, l^e\, \dot{\partial}_a K_{be}$.

It is noteworthy that a Finsler metric is one of scalar flag curvature iff

(3.14) $W^a{}_{bcd} = 0$.

Let us consider a mapping $\phi : F^n \rightarrow F^n$ and $\phi$ be diffeomorphism. Then $\phi$ is said to be a projective mapping if there exists a positive homogeneous scalar function P of degree one satisfying the relation

(3.15) $G^a = G^a + FP\, l^a$,

Wherein P is the projective factor [9].

Under a projective transformation with projective factor P, the Riemannian curvature tensor of Berwald connection change is given by the following expression

(3.16) $\check{K}^a{}_{bcd} = K^a{}_{bcd} + Fl^a\, \partial_b Q_{cd} + \delta^a{}_b Q_{cd} + \delta^a{}_c\, \partial_b Q_d - \delta^a{}_d \dot{\partial}_b Q_c$,

Wherein

(3.17) $Q_a = \lambda_a P - PP_a$

and

(3.18) $Q_{ab} = \dot{\partial}_a Q_b - \dot{\partial}_b Q_a$.

It is noted that if $Q_{ab} = 0$ then a projective transformation with projective factor P is said to be C-projective.

(3.19) $£_X G^a{}_{bcd} = \delta^a{}_b P_{cd} + \delta^a{}_c P_{bd} + \delta^a{}_d P_{cb} + Fl^a\, P_{bcd}$

and

(3.20) $£_X K^a{}_{bcd} = \delta^a{}_b (P_{d!c} - P_{c!d}) + \delta^a{}_d\, P_{b!c} - \delta^a{}_c\, P_{b!d} + Fl^a\, \dot{\partial}_b (P_{d!c} - P_{c!d})$.

Since

(3.21) $Q_{bc} = P_{b!c} - P_{c!b}$,

We have [5,7]:

(3.22) $\dot{\partial}_a P_{b!c} = P_{ab!c} - P_e G^e{}_{abc}$,

Contracting a and c in (3.21), yield

(3.23) $£_X K_{bd} = P_{d!b} - nP_{b!d} + F\, P_{bd!h}\, l^h$,

Hence, obtain

(3.24) $£_X (Fl^e\, \dot{\partial}_d K_{be}) = -(n-1)F\, P_{bd!h}\, l^h$,

Consequently yields

(3.25) $£_X (K_{bd} + \{1/(n+1)\}Fl^e\, \dot{\partial}_d K_{be}) = P_{d!b} - nP_{b!d}$

and

(3.26) $£_X (K_{db} + \{1/(n+1)\}Fl^e\, \dot{\partial}_b K_{de}) = P_{b!d} - nP_{d!b}$,

As a consequence of equations (3.25) and (3.26), we get

(3.27) $P_{b!d} = \{1/(1-n^2)\}£_X[K_{db} + \{1/(n+1)\}Fl^e\, \dot{\partial}_b K_{de} + nK_{bd} + \{n/(n+1)\}Fl^e\, \dot{\partial}_d K_{be}]$.

If $Q_{ab} = 0$, the equation (3.20) reduces to

(3.28) $£_X K^a{}_{bcd} = \delta^a{}_d\, P_{b!c} - \delta^a{}_c\, P_{b!d}$,

By virtue of equations (3.27) and (3.28), we obtain new type of tensor

(3.29) $\dot{W}^a{}_{bcd} = K^a{}_{bcd} - \{1/1-n^2\}[\delta^a{}_d\, \{\dot{K}_{bc} + \{n/(n+1)\}\, Fl^e(\dot{\partial}_c K_{be} - \dot{\partial}_b K_{ce})\} + \delta^a{}_c\{\dot{K}_{bd} + \{n/(n+1)\}Fl^e\, (\dot{\partial}_d K_{be} - \dot{\partial}_b K_{de})\}]$,

As a consequence of

(3.30) $l^a l^e\, \dot{\partial}_b K_{ae} = 0$

and

(3.31) $\dot{W}^a{}_c = \dot{W}^a{}_{bcd} l^b l^d$.

We obtain

(3.32) $\dot{W}^a{}_c = K^a{}_c - \{1/(1-n^2)\}[Fl^a\, \dot{K}_{0c} - \delta^a{}_c\, \dot{K}_{00}]$.

The tensor K is called C-projective Weyl curvature. If C-projective Weyl-curvature is vanish then a Finsler metric F is called C-projective Weyl metric.

In view of above discussions, we have the following theorems:

**Theorem 3.1:**

Let F be a C-projective Weyl metric. Then prove that F is a Weyl metric.

**Proof:**

Let us assume that

(3.33) $K^a{}_c - \{1/(1-n^2)\}[Fl^a\, \dot{K}_{0c} - \delta^a{}_c\, \dot{K}_{00}] = 0$,

Contracting (3.33) by $l^a$ yields

(3.34) $F^3\, \dot{K}_{0c} - l_c\, \dot{K}_{00} = 0$,

Consequently,

(3.35) $\dot{K}_{0c} = F^{-3}\, l_c\, \dot{K}_{00}$,

By virtue of equations (3.33) and (3.35), we obtain

(3.36) $K^a{}_c = \{1/(1-n^2)\}h^a{}_c\, \dot{K}_{00}$,

Consequently F is said to be scalar flag curvature. Therefore F is a Weyl metric.

**Theorem 3.2:**

Let F be a Finsler metric of scalar flag curvature κ. Then C-projective Weyl curvature is defined as $\dot{W}^a{}_c = (1/3)F^3 l^a\, \kappa_c$, wherein $\kappa_c = \dot{\partial}_c \kappa$.

**Proof:**

The Riemannian curvature of Berwald connection is given by[3]:

(3.37) $K^a_{bcd} = \kappa(\delta^a_c g_{bd} - \delta^a_d g_{bc}) + \kappa_b F(\delta^a_c F_d - \delta^a_d F_c)$
$+ (1/3)[F^2(h^a_c \kappa_{bd} - h^a_d \kappa_{bc}) + \kappa_d F(2\delta^a_c F_b - 2\delta^a_b F_c - g_{bc} l^a) + \kappa_c F(2\delta^a_d F_b - 2\delta^a_b F_d - g_{bd} l^a)$,

Wherein

(3.38) $\kappa_{bc} = \dot{\partial}_c \kappa_b$.

Consequently yields

(3.39) $K^a_b = \kappa F^2 h^a_b$,

As a consequence, we obtain [9]:

(3.40) $K_{bd} = (n-1)(\kappa g_{bd} + FF_d\kappa_b) + (1/3)(n-2)(F^2\kappa_{bd} + 2FF_b\kappa_d)$,

(3.41) $\dot{K}_{00} = (n^2-1)\kappa F^2$,

(3.42) $K_{0c} = (n-1)\kappa FF_c + (1/3)(n-2)F^2\kappa_c$

(3.43) $\dot{K}_{0c} = (n^2-1)(\kappa FF_c + (1/3)F^2\kappa_c$,

By virtue of equations (3.32), (3.39), (3.41) and (3.43), we obtain

(3.44) $\dot{W}^a_c = (1/3)F^3 l^a \kappa_c$.

**Theorem 3.3:**

If F is a Finsler metric of constant flag curvature with K = κ. Then F is C-projective Weyl metric.

**Proof:**

If F is of constant flag curvature κ then equation (3.37) reduces in the form

(3.45) $K^a_{bcd} = \kappa(\delta^a_c g_{bd} - \delta^a_d g_{bc})$,

Yields

(3.46) $K_{bd} = -(n-1)\kappa g_{bd}$,

and

(3.47) $\dot{K}_{bd} = -(n^2-1)g_{bd}$,

By virtue of equations (3.29), (3.46) and (3.47) yields

(3.48) $\dot{W}^a_{bcd} = 0$,

Consequently yields

(3.49) $\dot{W}^a_c = 0$.

In Finsler metric F of scalar flag curvature with dimension n≥3, we have a projective transformation with the projective factor P, we have the following conditions:

(3.50) $\dot{S}_{ab} = S_{ab} + \{(n+1)/2\}P_{ab}$,

and

(3.51) $\dot{S}_{eb}\dot{G}^e_a = S_{eb}G^e_a + PS_{ab} + \{(n+1)/2\}(P_{eb}G^e_a + PP_{ab})$.

In view of above discussion, we have the following:

**Theorem 3.4:**

τ-curvature is C-projective invariant.

**Proof:**

For a projective transformation, we have

(3.52) $\dot{\tau}_{ab} = F\dot{S}_{abc} l^c$,

By virtue of equations (3.50), (3.51) and (3.52) yields

(3.53) $\dot{\tau}_{ab} = \tau_{ab} + \{(n+1)/2\}(Fl^e\lambda_e P_{ab} - P_{eb}G^e_a - P_{ae}G^e_b)$,

Consequently follows

(3.54) $Fl^e\dot{\partial}_a Q_{bc} = Fl^e\lambda_e P_{ab} - P_{eb}G^e_a - P_{ea}G^e_b$,

From equations (3.53) and (3.54), we get

(3.55) $\dot{\tau}_{ab} = \tau_{ab} + \{(n+1)/2\}Fl^e\dot{\partial}_a Q_{be}$,

If we take C-projective mapping i.e. $Q_{ab} = 0$. Follows

(3.55) $\dot{\tau}_{ab} = \tau_{ab}$.

Hence, τ -curvature is C-projective invariant.

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